Mathematics of Voting – Weighted Voting Systems

While in political elections, we expect a one person—one vote situation, we may find a different situation in a company board room, a shareholder meeting, or in international organizations where some participants (we call them players) have more power than others.

**Language:**
- We have players $P_1, P_2, P_3$ etc…
- Each of them has a certain **weight** $(w_1, w_2, w_3, \ldots)$
- A certain number of votes, $q$, is needed to **pass a motion—the quota $q$.**

Our short-hand notation for voting systems, eg. $[17: 9, 6, 4, 2]$ lists the quota first (17) and then the weights of the four players, in decreasing order.

**Investigations:**

1) The ABCD corporation, which is valued at $20 million, is owned by four family members. The following table lists the share each of the family members, A, B, C, and D owns:

<table>
<thead>
<tr>
<th></th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\text{Total}$</td>
<td>$8 \text{ m}$</td>
<td>$6 \text{ m}$</td>
<td>$5 \text{ m}$</td>
<td>$1 \text{ m}$</td>
<td>$20 \text{ m}$</td>
</tr>
</tbody>
</table>

The company by-laws require consent from two thirds of the total capital to pass any motion.

a) What is the amount of shares needed to pass a motion?

b) How would you describe this voting system in short-hand notation?

2) Next, consider a company board with four members $P_1, P_2, P_3,$ and $P_4$. $P_1$ has 6 votes, $P_2$ has 5 votes, $P_3$ has 3 votes, and $P_4$ has one vote. A total of seven votes is needed to pass a motion.

a) What is the short-hand notation for this voting system?

b) Now, suppose that there is a motion to sell the company. Board members $P_1$ and $P_4$ vote to sell. Board members $P_2$ and $P_3$ vote not to sell. Does the motion pass?

c) What is surprising about this outcome?
d) Let’s find out what has gone awry. What is the total number of votes of all four board members?

e) How does the quota compare to that?

f) What restriction do you suggest we put on the quota, so that it leads to a meaningful voting system?

3) So, consider again the previous board room situation, and pick a new quota. Suppose we choose a quota of 17 votes, so we get the voting system [17: 6, 5, 3, 1].

a) How can this motion pass, or be defeated? Is there a winning coalition?

b) What restriction do you suggest we place on the quota, in order to obtain a meaningful voting system? Explain.

4) Next, consider the following voting system: [10: 3, 3, 3, 3, 3]. All players are equal.

a) What is the total number of votes?

b) Does the quota satisfy the restrictions you suggested in the previous problem?

c) How many players are needed to pass a motion?

d) Each player has 3 votes. How does this voting system compare with a one person—one vote situation, such as the voting system [3: 1, 1, 1, 1, 1]? Explain.
5) Next, consider the voting system [15: 5, 4, 3, 2, 1].

a) What is the total number of votes?

b) How does this compare to the quota?

c) Which of the players seems to be the most powerful?

d) How many people are needed to pass a motion?

e) What would be a good name for this voting system?

f) How does this voting system differ from the *one person—one vote* voting system [5: 1, 1, 1, 1, 1]?

g) Which of the players of [15: 5, 4, 3, 2, 1] is therefore *really* the most powerful?

6) Consider [11: 12, 5, 4].

a) Which player seems to be the most powerful?

b) How many players are needed to pass a motion?

c) In view of your answer in (b), what would be a good name for the first player?

d) What would be a good name for the other players?

7) Consider [12: 9, 5, 4, 2]

a) Which player seems to be most powerful?

b) What players are needed to pass a motion?

c) How would you describe the power that the first player has? (think of the U.N. Security Council)

8) Finally, consider the XYZ corporation. Mr. X has 99 votes, Mrs. Y has 98 votes, and Ms. Z has a measly 3 votes. Decisions are made by majority rule.

a) What is the total number of votes?
b) What is the quota? (And does it satisfy the restrictions you suggested?)

c) Write down the short-hand notation of this voting system.

d) Who seems to be the most powerful player?

e) Who seems to be the least powerful player?

f) How many players, at least, are needed to pass a motion?

g) How does this voting system compare to the voting system [2: 1, 1, 1]?

h) Which of the players, if any, do you now consider to be the most powerful?