

Fractals, Pascal's Triangle, and the p -adic Numbers

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[Start by putting the following sequences on the board:]

2, 7, 57, 182, 2057, 14 557, 45 807, 280 182, 6 139 557, 25 670 807, 123 327 057, 123 327 057, ...

3, 18, 68, 443, 1068, 1068, 32 318, 110 443, 1 672 943, 3 626 068,

23 157 318, 120 813 568, 1 097 376 068, 1 097 376 068, ...

What do these sequences converge to? Every calculus student should guess that they diverge. We'll see how they can be seen to converge to $\pm i$ (where $i = \sqrt{-1}$).

We'll start with Fractals, as the title suggests.

Ask audience for informal definition of a fractal. Two definitions I am aware of, and looking for are:

- An object that is self-similar; that is, an object that has a part of itself that contains a smaller copy of the whole.
- An object that has a fractional dimension. Fractal, as a term, is an abbreviation of this definition.

One way to compute the dimension of a fractal is to compute the Self-Similarity Dimension:

$$\dim(X) = \frac{\ln(\text{Number of Copies})}{\ln(\text{Length Scaling Factor})}$$

and, if $\dim(X)$ is not a whole number, then X is a fractal.

- EXAMPLE: Consider a line segment. If you break a line segment into two pieces, each of which is half of the length of the whole segment, your number of copies is 2, the Length Scaling Factor is 2, and so we get $\dim(\text{line segment}) = \frac{\ln(2)}{\ln(2)} = 1$, as expected.

- **EXAMPLE:** Consider the Cantor Ternary Set. This is composed of two copies, each of which is $1/3$ of the length. This gives $\dim(\text{Cantor Set}) = \frac{\ln(2)}{\ln(3)} \sim .6309$ and so the Cantor Set is a fractal. Its dimension is between 0 and 1, which matches our intuition that it is more than a point, but less than a line.
- **EXAMPLE:** Consider an equilateral triangle. This can be broken up into four sub-triangles, each of which has side length $1/2$ of the original triangle. So, we get $\dim(\text{Triangle}) = \frac{\ln(4)}{\ln(2)} = 2$.
- **EXAMPLE:** Consider the Sierpinski Triangle. This can be broken into three copies, each of which has side length $1/2$ of the original triangle. So, we get $\dim(\text{Sierpinski Tri}) = \frac{\ln(3)}{\ln(2)} \sim 1.585$.
- **EXAMPLE:** Consider the Cube. This can be broken into 8 copies of itself, each of which has side length $1/2$ of the original cube. This gives $\dim(\text{Cube}) = \frac{\ln(8)}{\ln(2)} = 3$.
- **EXAMPLE:** Consider the Menger Sponge. This has 20 copies, each of which has side length $1/3$ of the original, and so we get $\dim(\text{Menger Sponge}) = \frac{\ln(20)}{\ln(3)} = 2.7268$.

[Now, put up on the screen Pascal's Triangle (mod 2), zoomed way out. Notice similarity to Sierpinski's Triangle. Zoom in, realize what we've got. Now, zoom back out, switch to Pascal's Triangle (mod 3), get $\dim(X_3) = \frac{\ln(6)}{\ln(3)} \sim 1.6309$; go to (mod 5), get $\dim(X_5) = \frac{\ln(20)}{\ln(5)} \sim 1.8614$.]

In general, we get $\dim(X_p) = \frac{\ln(\frac{1}{2}p^2 + \frac{1}{2}p)}{\ln(p)} = 1 + \frac{\ln(\frac{p+1}{2})}{\ln(p)} = 1 + \frac{\ln(p+1) - \ln(2)}{\ln(p)}$. The second term converges to 1 from below and is always increasing for $p \geq 2$, so the dimensions are always increasing and between 1 and 2, which matches our intuition.

Now, consider the p -adic integers. These are formed as follows: For any usual integer, we write it as a base p integer, and these digits form its p -adic expansion. It is customary to write it in reverse order as follows:

EXAMPLE: $432 = 3 \times 5^3 + 2 \times 5^2 + 1 \times 5^1 + 2 \times 5^0$; we would normally write this as $(3212)_5$, but here we write it as $(2, 1, 2, 3, 0, 0, 0, \dots)$.

Now consider the equation $bx \equiv a \pmod{p}$. This has a unique solution whenever $b \not\equiv 0 \pmod{p}$; then, if we follow this with $bx \equiv a \pmod{p^n}$ for $n = 1, 2, 3, \dots$, we get a series of solutions which represents the fraction $\frac{a}{b}$:

EXAMPLE: To write $-\frac{1}{2}$ as a 5-adic integer, we solve:

$$2x \equiv -1 \pmod{5}, \text{ which gives } x \equiv 2$$

$$2x \equiv -1 \pmod{25}, \text{ which gives } x \equiv 12 = 2 * 5^0 + 2 * 5^1$$

$$2x \equiv -1 \pmod{125}, \text{ which gives } x \equiv 62 = 2 * 5^0 + 2 * 5^1 + 2 * 5^2$$

and so on, yielding a 5-adic expansion for $-\frac{1}{2} = (2, 2, 2, 2, \dots)$.

Now consider the equation $x^2 + 1$. We know that $x^4 - 1$ has four distinct solutions (mod 5) by Fermat's Little Theorem. The solutions $x \equiv \pm 1$ solve the factor $x^2 - 1$, and the solutions $x \equiv 2, 3$ solve the factor $x^2 + 1$. If we continue to solve the equation $x^2 + 1 \pmod{5^n}$ increasing powers of 5, we get pairs of solutions, one congruent to 2 mod 5, and the other

congruent to 3 mod 5, yielding the sequences we started with, so i and $-i$ are both 5-adic integers. Their briefer expansions are:

$$(2, 1, 2, 1, 3, 4, 2, 3, 0, 3, 2, 2, 0, \dots)$$

$$(3, 3, 2, 3, 1, 0, 2, 1, 4, 1, 2, 2, 4, 0, \dots)$$

Fun Facts about the p -adic integers:

- They have a natural metric, where $d(a, b) = \left(\frac{1}{p}\right)^n$, where n = length of initial string of digits in the p -adic expansion that agree. So, for example, looking at the expansions for i and $-i$, above, have distance 1, since $n = 0$; they differ in their first digit. The first one has distance $\frac{1}{5}$ from $-\frac{1}{2}$, since they agree in their first digit but not the second. The distance between i and 432 is $\frac{1}{125}$. This distance measures the level of p -divisibility of the difference $a - b$.
- The p -adic integers can be expanded to form the p -adic numbers by adjoining $\frac{1}{p}$, and taking the closure under addition and multiplication. This makes the expansion infinite in both directions, and allows for the possibility that the n in the metric could be negative.
- The p -adic numbers, under this metric, are the only other possible completion of the rational numbers besides the reals.

Now, we'll try to compute the fractal dimension of the 3-adic integers. Start with a picture:

[Put three points in a triangle.]

These are 0, 1 and 2 – they are 1 unit apart from each other, since they are all different in the first digit. Then, 3 and 6 are distance $\frac{1}{3}$ from 0; 4 and 7 are $\frac{1}{3}$ from 1, and 5 and 8 are $\frac{1}{3}$ from 2.

[These six points form mini-triangles, at $\frac{1}{3}$ scale below each of the three initial points. Repeat with the integers 9 through 26, each of which is $\frac{1}{9}$ from one of the 9 existing points. This picture lies a bit: e.g., 0 is distance 1 from every point that is around 1 and also every point around 2.]

Fractal dimension turns out to be 1: three copies, each at $\frac{1}{3}$ scale. Same is true for p -adics for any other p .