No Cell Phone Signal?
Algebraic Topology to the Rescue!

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The problem we are considering is the work of Vin de Silva and Robert Ghrist. This talk is largely based on their article in the Notices of the AMS, January 2007 issue.

First, what is algebraic topology? Topology is an abstract study of the concept of shape. Algebraic topology is the branch of mathematics that uses algebraic techniques to find out information about the shape of topological objects. The problem we'll consider today is an example of how linear algebra can be applied to a practical topology problem.

Why am I interested in this problem? In my first year of graduate school, I took an introductory course in algebraic topology, where we did exactly what I just described – use linear algebra to distingush among different topological objects. At the time, I found it to be very abstract, and not very intuitive. I decided that I would not study any more of it once the course was over. After getting talked into another semester of algebraic topology, I never looked back, and found it to be beautiful, intuitive, and fascinating. I wrote my dissertation in that field. When I read the article I am presenting on, I finally found a fully practical use for this too-abstract, not-intuitive-enough field. Wrong once again.

The problem we will consider is this:

ASSUMPTIONS:

- Assume we have a region, that can take any (reasonable) shape, and we have a collection of 'towers', each of which can be located anywhere in the region.
- For each tower, there is a circular neighborhood (that contains the tower) that the tower can provide a signal to.
- Any tower can communicate with any other tower that is contained in its neighborhood (or possibly a bit larger; up to $\sqrt{3}$ times larger).
- The boundary of the region is completely covered by a chain of towers, each of which can communicate with its two nearest neighbors.

THE PROBLEM: Is there any part of the region that does not have a tower providing a signal to it?

Strangely enough, there is a branch of algebraic topology that is perfectly designed to answer this kind of question. It works roughly like this:

[Draw a picture of an annulus.]

This is a geometric figure called an annulus. What feature would you say is most noticable about it?

[Looking for "It's got a hole in it."]

[Draw simplified picture of Annulus, triangulated. Label edges $e_1, \ldots, e_9$. Direct them. Point out how it not quite an annulus. Make it really an annulus. Write some equations of some cycles. Write some equations of some boundaries. Show how some cycles can get reduced to nothing. Show how the others all reduce to the same thing, the equation for the hole, including the outer boundary.]

So, it should seem convincing that one can detect a 'hole' algebraically by detecting an equation in the edges that cannot be reduced to zero, and we can detect its location within the complex by its equation. If there were more holes, we would be able to detect them as well by finding additional equations that also cannot be reduced to zero.

To apply this idea to the problem at hand, we start by drawing a graph; a graph in the technical sense of graph theory. We put a vertex somewhere for each tower, and then, if two towers have overlapping neighborhoods that allow them to communicate with each other, we connect them with an edge. Notice that we don't have to put the vertices in a position that makes sense with its actual location, we just need to make sure that any two towers that can speak to each other have their vertices connected by an edge. If there are three neighborhoods that overlap, we connect the three vertices with a triangle. If four neighborhoods overlap, then we connect all four with a tetrahedron, and so on with higher and higher dimensional triangular objects. (These are formed by "taking the cone" over the previous one – start with a line segment, pick any point off the segment, connect every point in the segment with the new point off the segment. This yields a triangle. Pick any point off the triangle, connect every point of the triangle to the new point, get a tetrahedron. Repeat.)

Once we've pasted in all of these triangular objects, this is called the Vietoris-Rips Complex.

Now, we compute whether or not the outer boundary can be reduced to zero or not. If it can, we have complete coverage; if it can't, there will be nontrivial cycles, and we will know which towers are nearest the gap in coverage.

This is useful, but, in fact, we can do even better.

[Ask the audience where we used the fact that we knew where the towers were located. Never, just need to know that we have an unbroken cycle around the boundary.]

So, the same computation could be applied to simple sensors that we don't ever really know where they are, say, if they were dropped from an airplane onto a minefield. They don't even have to stay in one place – we just have to re-compute the equations regularly with updated connection information to keep informed about possible gaps in coverage, so we could, for example, do the same thing with mines in a bay, where the water currents will move your sensors around.

These ideas can also be used if you drop additional sensors onto the field; you can detect which sensors are essential to maintaining effective coverage, and then you can power down some, and trade 'live' and 'resting' sensors to save on battery power.