Topology Seminar, part II

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5 Constructing the Iwasawa Invariants

Here, we are following [Iwa72], [Iwa58], and [FW79].

Given a character χ on $(\mathbb{Z}/n\mathbb{Z})^*$, the number of elements in the image of χ is called the *conductor* of χ, and is usually denoted f_χ, or simply f. A character χ is called *even* if f_χ is an even number, and likewise for odd. We write $\delta_\chi = +1$ if χ is odd, and $\delta_\chi = 0$ if χ is even. It is easy to show that $\chi(-1) = \delta_\chi$ for all characters χ. If $f = 1$, then χ is the trivial character χ_0.

Definition 5.1 *The* generalized Bernoulli Numbers associated to the character χ *are generated by the following generating function (where f is the conductor of χ):*

$$F_\chi(t) = \sum_{a=1}^{f} \frac{\chi(a) t e^{at}}{e^{ft} - 1} = \sum_{n=0}^{\infty} B_{n,\chi} \frac{t^n}{n!}$$

Notice that $B_{n,\chi_0} = B_n$. If χ is non-trivial, then $B_{0,\chi} = 0$, $B_{n,\chi} = 0$ when $n \not\equiv \delta_\chi$, and $B_{n,\chi} \neq 0$ whenever $n \equiv \delta_\chi$.

We can use these to construct the Dirichlet $L-$functions, which are the corresponding generalization of the Riemann $\zeta-$function:

Definition 5.2 *The* Dirichlet $L-$functions *are defined via:*

$$L(s, \chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}$$

Just as

$$\zeta(s) = \prod_{p \text{ is prime}} (1 - p^{-s})^{-1}$$

the Dirichlet $L-$functions satisfy:

$$L(s, \chi) = \prod_{p \text{ is prime}} (1 - \chi(p) p^{-s})^{-1}$$

and also $L(1 - n, \chi) = -B_{n,\chi}/n$ for $n \geq 1$.

Computing $L(1, \chi)$ for χ nontrivial was a difficult problem solved in 1952 by H. Hasse.

Iwasawa constructed a similar collection of functions defined on the $p-$adic numbers, which are called the $p-$adic $L-$functions, and are denoted $L_p(s, \chi)$.

He used these to construct the Iwasawa invariants, which I will now consider.

Recall that h_p is the class number of the cyclotomic field $\mathbb{Q}(\zeta_p)$. The field $\mathbb{Q}(\zeta_p)$ contains a maximal real subfield, which is equal to $\mathbb{Q}(\zeta_p + {\zeta_p}^{-1})$. The class number of this is denoted ${h_p}^+$. Kummer showed that ${h_p}^+$ divides h_p; the quotient is denoted ${h_p}^-$; the factor ${h_p}^-$ is called the *first factor* of the class number h_p, and ${h_p}^+$ is called the *second factor*.

Kummer showed that a prime p divides h_p if and only if p divides the first factor h_p^-. Recall that a Bernoulli number is regular if and only if it does not divide the class number h_p, and so therefore its first factor h_p^-.

Let $q_n = p^n$, and consider the cyclotomic fields $\mathbb{Q}(\zeta_{q_n})$. Let $h_{q_n}^-$ denote the first factor of the class number of $\mathbb{Q}(\zeta_{q_n})$, and let p^{e_n} be the largest power of p that divides $h_{q_n}^-$.

Theorem 5.3 (Iwasawa) *There exist constants λ_p, μ_p, and ν_p that are independent of n for sufficiently large n such that:*

$$e_n = \lambda_p n + \mu_p p^n + \nu_p$$

Based on the proof, it seems very likely to me that this can be strengthened to $n \geq 0$ if we could prove even a weak bound on the index of irregularity of p relative to the size of p.

This can be easily generalized to an arbitrary integer $q_n = mp^n$ with m coprime to p, but I am particularly interested in the case $q_n = p^n$.

The constants λ_p, μ_p, and ν_p are called the *Iwasawa invariants.* There are a few results known about the Iwasawa invariants:

Theorem 5.4 (Iwasawa) *If p is regular, all of the Iwasawa invariants are zero.*

Theorem 5.5 (Iwasawa) *If p is irregular, either λ_p or μ_p is nonzero.*

Theorem 5.6 (Ferrero and Washington) *$\mu_p = 0$ for all p*

Corollary 5.7 *$\lambda_p \neq 0$ for all irregular primes p.*

I worked through some of Iwasawa's papers to see what results, if any could be gleaned regarding the $p-$divisibility of the Bernoulli numbers from Ferrero and Washington's theorem, and the best I came up with is:

Theorem 5.8 *One of the following sequence of congruences does not hold:*

$$(-1)^n \frac{B_n}{n} \equiv 0 \pmod{p}$$

$$(-1)^n \frac{B_n}{n} \equiv (-1)^{n+(p-1)/2} \frac{B_{n+(p-1)/2}}{n+(p-1)/2} \pmod{p^2}$$

$$(-1)^n \frac{B_n}{n} \equiv (-1)^{n+(p-1)} \frac{B_{n+(p-1)}}{n+(p-1)} \pmod{p^3}$$

$$\vdots$$

If p is irregular, the first holds, but there is no certainty about which of the congruences after that does not hold.

The best that I know of that is known about the values of λ_p and ν_p for irregular p is the following:

Theorem 5.9 (Washington? – it is given in his text [Was82]) *Let p be an irregular prime. If:*

1. *(Kummer-Vandiver Conjecture) p does not divide the second factor of the class number h_p^+.*

2. *For any index j where $p | B_n$, we have that $\frac{B_n}{n} \not\equiv \frac{B_{n+p-1}}{n+p-1} (\mathrm{mod} p^2)$.*

3. *For the generalized Bernoulli number associated to the Teichmüller character, $B_{1,\omega}$, we have that $B_{1,\omega} \not\equiv 0 (\mathrm{mod} p^2)$.*

then $\lambda_p = \nu_p = i(p)$, where $i(p)$ is the index of irregularity of p, so that $ord_p(h_{p^n}) = i(p)(n+1)$.

This led me back to topology.

6 Profinite Completion of a $CW-$complex

Here I am primarily following [Sul05], and my own work.

I decided to see if any of the existing maps that we know from the topology of part I could be extended to give maps defining any additional number-theoretic structure other than the Bernoulli Numbers and $2^{2n-1}-1$. This would require a change in perspective from the $p-$local to the $p-$adic, and a corresponding change in the spaces that I am looking at.

In the $p-$local setting, there is a clear $CW-$complex construction that gives an effective road map for the algebraic topology of the localized spaces. In the $p-$adic setting, we are not quite so lucky. We'd like a construction that will induce tensor product with the $p-$adic integers in each of homotopy, homology, and cohomology. However, the proof is more of an existence proof, and the results not quite as carefully understood. The process is called *profinite completion.* There were two distinct approaches to it when it was first constructed, one due to Michael Artin and Barry Mazur, the other due to Aldridge Bousfield and Daniel Kan. Sullivan followed Artin and Mazur, and I (perhaps unwisely) have followed his lead. Artin and Mazur's construction leads only to an object that is the universal object in a categorical construction, with no claim regarding its realization as a topological space. Sullivan sought to lay out conditions that would guarantee that the completion would be realizable as a $CW-$complex.

My (partly completed) goal is to construct $\widehat{BU}_p$ and $\widehat{BO}_p$, compute their homotopy, homology, and cohomology groups, and show that the $p-$local constructions all hold in the $p-$adic setting, showing that no information is lost in this translation. The next step is to try to get additional information from the properties of the $p-$adic setting, particularly the addition of the $(p-1)^{\mathrm{st}}$ roots of unity, and therefore, one hopes, the addition of certain Dirichlet characters, their associated Bernoulli numbers, and more. It is my conjecture that the $J-$homomorphism and/or the closely related Adams-Bott homomorphism can be extended to construct a map which induces multiplication by the generalized Bernoulli numbers associated to the Teichmüller character, and possibly others.

Sullivan states the following theorems:

Theorem 6.1

$$\pi_n(\widehat{X}_\ell) \cong \widehat{\pi_n(X)}_\ell$$

where $\ell = \{$*all primes*$\}$.

Theorem 6.2

$$H^n(\widehat{X}_\ell; \widehat{\mathbb{Z}}_\ell) \cong \widehat{H^n(X;\mathbb{Z})}_\ell$$

where $\ell = \{$*all primes*$\}$.

He then follows this, several pages later, with an example, due to Bousfield, that is constructed as follows:

Let S^n be the $n-$sphere, with n odd, $n > 1$. Then $\pi_n(S^n) \cong \mathbb{Z}$, and $\pi_k(S^n)$ for $k > n$ are all finite abelian groups. If consider the rationalization of S^n (which can be seen as the localization at $\emptyset$, and induces $\otimes\mathbb{Q}$ on all homotopy, homology, and cohomology groups), this has $\pi_n\left((S^n)_{(\emptyset)}\right) \cong \mathbb{Q}$, and all other dimensions trivial. Similarly, if we consider the rationalization of $\widehat{S^n}_\ell$, this, by 6.1, has $\pi_n\left(\left(\widehat{S^n}_\ell\right)_{(\emptyset)}\right) \cong \widehat{\mathbb{Z}}_\ell$, and all other homotopy groups trivial. This makes $\left(\widehat{S^n}_\ell\right)_{(\emptyset)}$ into a $K(\widehat{\mathbb{Z}}_\ell, n)$. In this case, we view $\widehat{\mathbb{Z}}_\ell$ as a vector space over $\mathbb{Q}$ with uncountably many dimensions.

We then use Serre's theorem on the homology of Eilenberg-MacLane spaces to show that the rational homology of a $K(\widehat{\mathbb{Z}}_\ell, n)$ is an exterior algebra with generators equal to a basis of the vector space in dimension n. Since this has uncountably many distinct generators, there are also uncountably many generators in all dimensions that are a multiple of n. Then, since this is true for the rationalization, the integral homology of the non-rationalized space also has uncountably many generators in all dimensions that are a multiple of n. By Universal Coefficients, we also have these generators in cohomology. So, by this calculation, we see that

$H^{kn}(\widehat{S^n}_\ell) \not\cong 0$ for all $k \geq 1$. However, Theorem 6.2 says that $H^{kn}(\widehat{S^n}_\ell) \cong \widehat{H^{kn}(S^n;\mathbb{Z})}_\ell$, but the right-hand side is zero for $n > 1$.

Hence, one of the following is incorrect:

1. Sullivan's Theorem 6.1
2. The localization construction, and its results on homotopy, homology and cohomology of the result.
3. Serre's computation of the homology of Eilenberg-MacLane spaces.
4. Sullivan's Theorem 6.2

It is certain that it is either 1 or 4 or both; it seems most likely, partly through private correspondence with Prof. Bousfield, that the latter is where the problem resides.

References

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