

Palindromes in Mathematics and Music

Prof. Christine von Renesse,
Jeffrey Vanasse, Kathleen Layton,
and the Math Exploration class, fall 2008
Department of Mathematics
Westfield State College, Westfield, MA 01086
cvonrenesse@wsc.ma.edu

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Fall semester 2008 I, Christine von Renesse, taught *Math Exploration: Mathematics and Music* at Westfield State College for the first time. The audience was 25 music majors who had to fulfill their mathematics requirement but also 2 mathematics majors and 5 other majors. The structure was set up in a very open inquiry-based style, with students in charge of classroom decisions such as time spend on a topic, and exploration questions. In this article I would like to share the process and some of the results of our palindrome exploration.

We started out exploring rhythms. My first task for my students was to find out how many different rhythms there are given a specific number of counts (for instance 8 eighths in a measure) and a number of beats (for instance 3 hits on a drum during those 8 eighths). After spending some time on the notation we agreed to write a rhythm in two different ways that seemed appropriate for our non-music majors who might struggle with the music standard notation.

1. Binary notation, for instance 10110110, with 1s as beats and 0s as off-beats.
2. Geometric notation, where we notate the counts around a circle and mark the beats as dots. If you connect the dots you can use the polygon to visualize the rhythm.

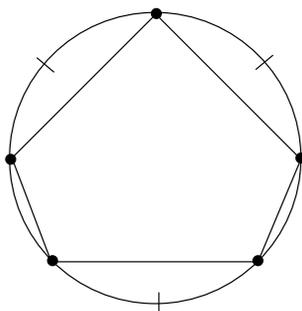


Figure 1: A Geometric Palindrome with 8 Counts and 5 Beats

During our work on counting rhythms we discovered Pascal's triangle, which was unknown to the non-mathematics majors. The number of rhythms given n counts and k beats can be found in row $n+1$ and entry $k+1$ in Pascal's triangle, see figure (2). For example, the box in the 5th row and 3rd entry containing a 6, tells you that there are 6 different rhythms given 6 counts and 4 beats. If you are familiar with binomial coefficients, you can compute the number of rhythms as

$$\binom{n+1}{k+1}.$$

The next challenge was to find and count rhythms that are palindromes. You may be familiar with the notion of a palindrome from words like ANNA which read the same forwards and backwards. Godfried Toussaint [1] describes

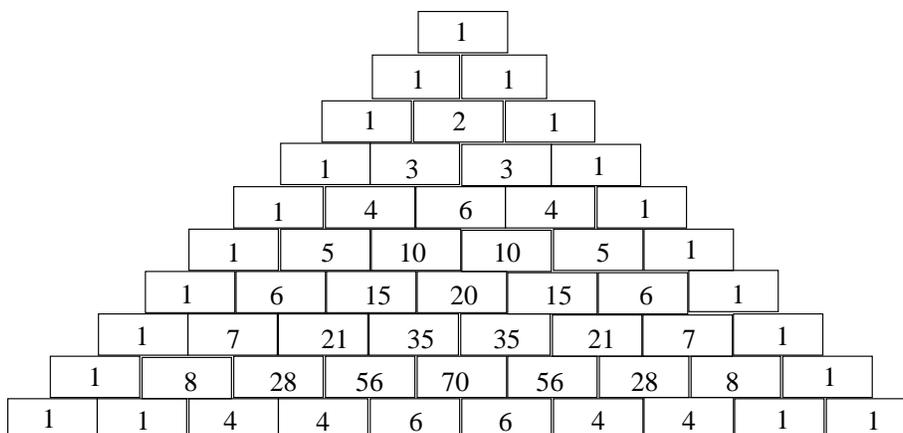


Figure 2: Pascal's Triangle

a palindrome as a rhythm that sounds the same if you play it forward or backward in the circle notation. Figure (1) shows a palindrome in that sense: If you play to the right, starting at the top, you get 10110110, if you play to the left you get 10110110.

Here my students came up with a startling realization: In binary notation 10110110 doesn't look like a palindrome at all! This wonderful confusion led to two weeks of intensive exploration. We called the binary version a musical palindrome and the geometric version a geometric palindrome and started comparing and counting them.

The following results are my students' work. The mathematics majors helped greatly with the notation and the precision of the argument. Enjoy!

Two Types of Palindromes

We noticed that there are two types of palindromes that disagree! To better understand what was happening we recorded the number of geometric palindromes and the number of musical palindromes in two triangles, similar to Pascal's triangle. The number of palindromes given n counts and k beats can be found in row $n + 1$ and entry $k + 1$, where $k, n \in \{0, 1, 2, \dots\}$.

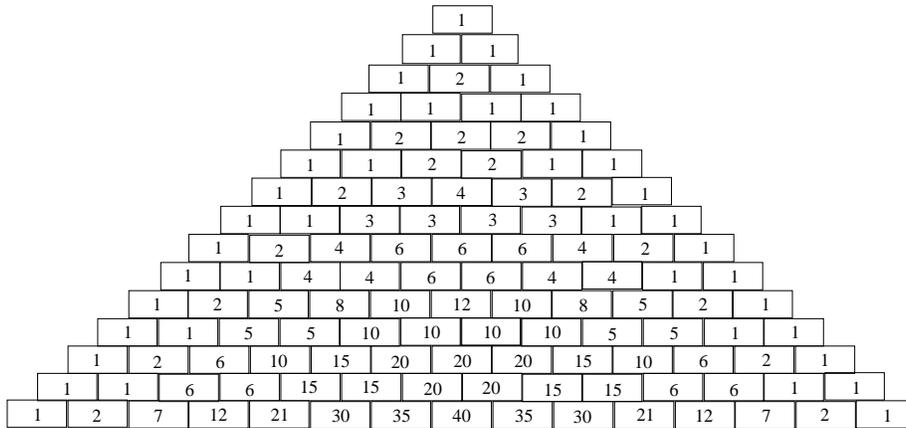


Figure 3: Geometric Triangle

The first triangle, see figure (3), we call the *geometric triangle*. Similar to Pascal's triangle, in some places two adjacent numbers add up to the number below. The same is true for the next triangle, see figure (4), which we call the *musical triangle*. The most noticeable trait of the musical triangle is the zeros in the rows that represent an even number of counts, in places with an odd number of beats.

There is an astounding number of patterns in the triangles, looking at the rows, columns and diagonals. For instance, notice the triangles inside the mu-

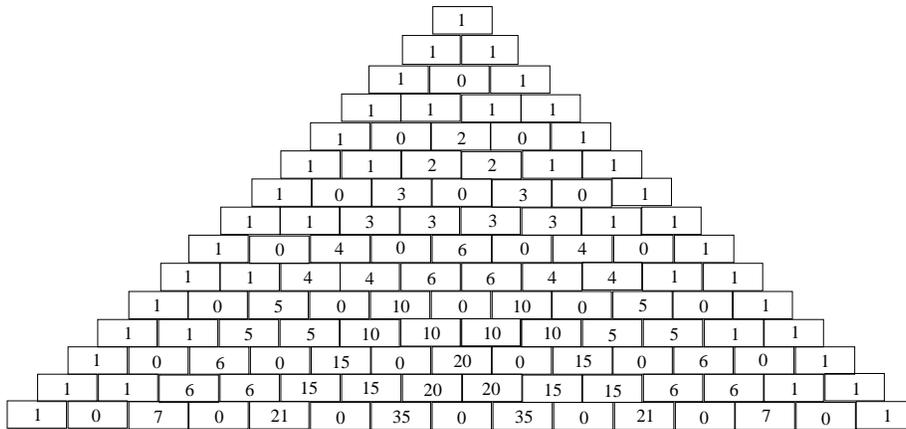


Figure 4: Musical Triangle

sical triangle, that contain 3 numbers but no zeros. Those triangles contain exactly the numbers from Pascal's triangle, see figure (5)!

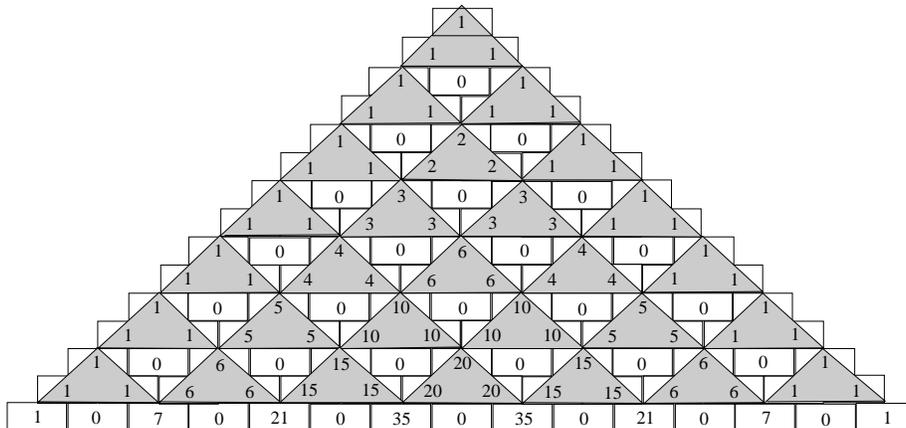


Figure 5: Pascal's Triangle inside the Musical Triangle

The Addition Pattern

One of the most interesting patterns in the triangles is the addition pattern, very similar to the one in Pascal's triangle. We discovered the formal proofs during class but would like to present here some examples and diagrams instead. We believe that this will give more insight to the reader. The formal proofs require the use of binomial coefficients and symmetry arguments, please contact us if you are interested in the details.

The following diagram, figure (6), shows a small section of the musical triangle, from rows 5 and 6. You can see how the number of musical palindromes in the upper row add up to the number of palindromes underneath: $1+2=3$. To get a musical palindrome in the 6th row you need to add either a beat or a pause to each palindrome in the upper row. In order to keep the symmetry of a musical palindrome you have to add the extra beat or pause in the center.

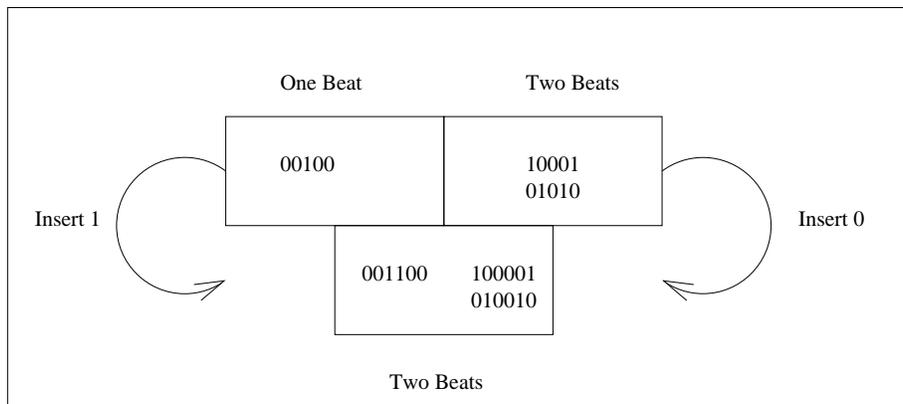


Figure 6: Addition Pattern in the Musical Triangle

So, why does the addition pattern fail sometimes? If there is an even number of beats it is impossible to insert a 1, and if there is an odd number of beats it is impossible to insert a 0 while keeping the symmetry of a musical palindrome. See figure (7).

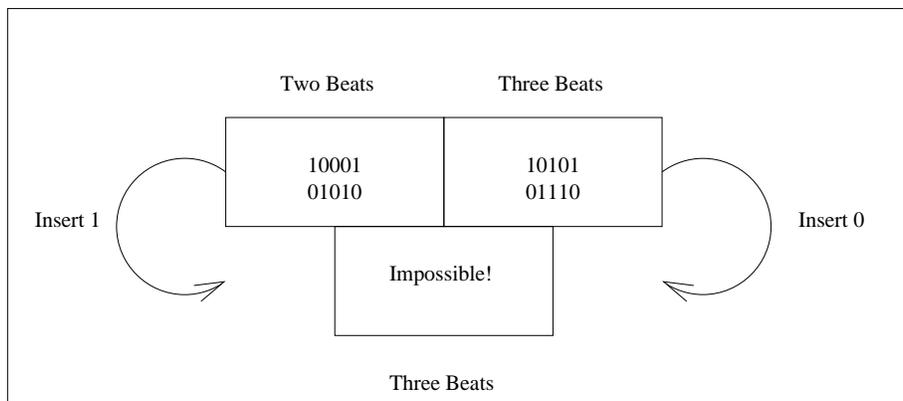


Figure 7: Addition Pattern in the Musical Triangle Fails

For the geometric triangle, figure (8) shows how the addition works: $2+2=4$. Again, we insert a beat or an off-beat in the center (here the bottom of the circle).

Figure (9) shows why the addition pattern can fail. When there is an even number of counts and an odd number of beats and you insert an off-beat you can get the same palindrome as if you are taking a geometric palindrome with one less beat (now zero or even number of beats) and add a beat.

Connections

Both triangles, the musical and the geometric, coincide in most entries. So it should be possible to find a “generic” way to make a geometric palindrome into a musical palindrome and vice versa. We discovered that you can take a musical palindrome, center it above the circle and then “wrap it around”. You will indeed get a geometric palindrome. See figure (10). If you have an even number of counts you need to split off one of the middle counts and add it to the bottom of the circle.

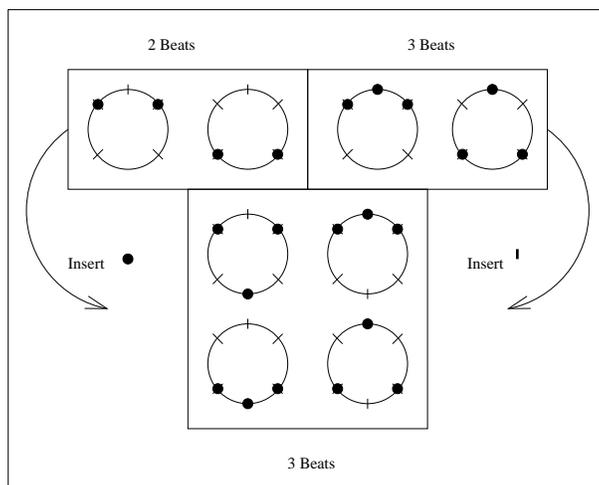


Figure 8: Addition Pattern in the Geometrical Triangle

On the other hand, starting with a geometric palindrome and “unwrapping” it, you will have a problem if there is an even number of counts: which way is the bottom number supposed to go? If you have an even number of beats, you can modify the procedure: take out the bottom point and add it back into the center of the musical palindrome. If there is an odd number of beats, we will not find a corresponding palindrome. Luckily, this matches the zeros in our triangle. See figure (11).

Weak Palindromes and Open Questions

For a rhythm to be a geometric palindrome, the polygon inside the circle needs to be symmetric with respect to the axis through the top and the bottom of the circle. Now, if we have a rhythm with a polygon that has reflectional symmetry but with respect to another axis, we call it a *weak geometric palindrome*. See figure (12). Thus any weak geometric palindrome can be rotated around the

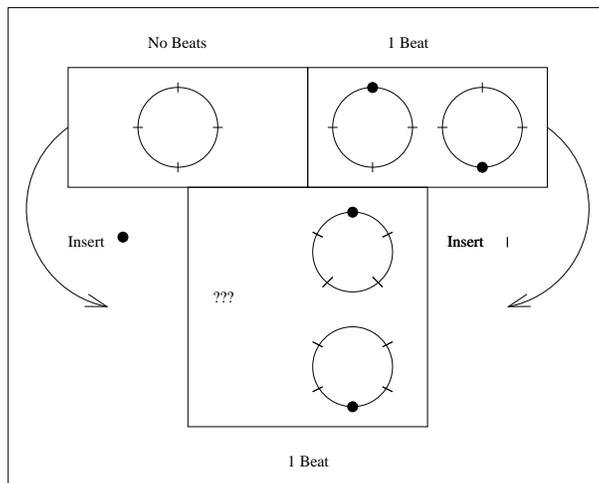


Figure 9: Addition Pattern in the Geometrical Triangle Fails

circle to form a geometric palindrome as defined above.

Similarly we call a rhythm a *weak musical palindrome* if it can be shifted to a musical palindrome as defined above. For instance 0011 can be shifted to 1001.

It is an open question at this point how the number of weak palindromes relates to the number of palindromes and if the corresponding triangles yield interesting patterns.

References

- [1] G. Toussaint, A Mathematical Analysis of African, Brazilian and Cuban Clave Rhythms, *Proceedings of BRIDGES: Mathematical Connections in Art, Music and Science*, Towson, MD, 2002.

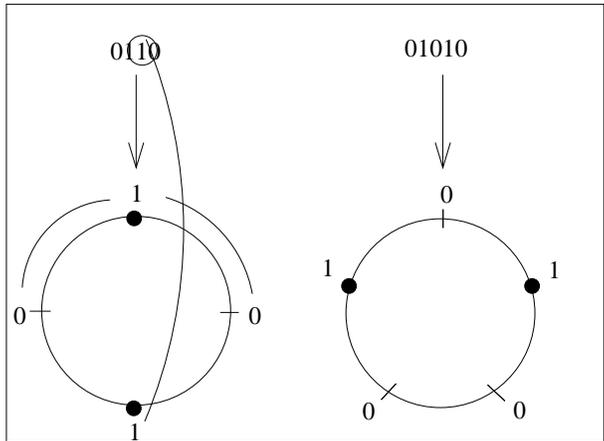


Figure 10: Wrapping a musical Palindrome around the Circle

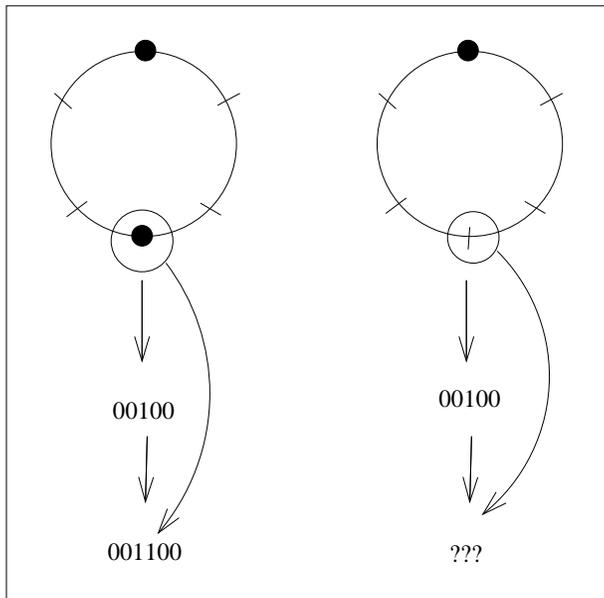


Figure 11: Unwrapping a geometric Palindrome fails

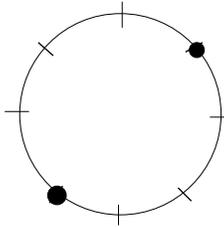


Figure 12: Example of a weak geometric Palindrome